where $\mathbf{v} = \dot{\mathbf{x}}(t)$. By choosing $t = s$, where s is the distance travelled by the particle, so $ds^2 = d\mathbf{x} \cdot d\mathbf{x}$, the velocity $\mathbf{v} = d\mathbf{x}/ds$ has speed $\mathbf{v} \cdot \mathbf{v} = 1$. Then, Eq. (C8) is identical to the equation of motion Eq. (C6).

Imagine the vector field passing through an initial surface, labeled by coordinate $s = 0$ and proceeding onward. Since each member of the family of subsequent s = constant surfaces is perpendicular to $\hat{\mathbf{v}}$, we have $\nabla s(\mathbf{x}) = C(\mathbf{x})\hat{\mathbf{v}}(\mathbf{x})$, where C is some function. However,

$$\hat{\mathbf{v}} \cdot \nabla s = \sum_i \frac{dx^i}{ds}\frac{\partial}{\partial x^i} s = \frac{ds}{ds} = 1,$$

so $C = 1$ and $\nabla s = \hat{\mathbf{v}}$.

Then, Eq. (C7) may be written:

$$\mathcal{A}_0(\mathbf{x}) \equiv \int_0^{s(\mathbf{x})} ds \mathcal{L}(\mathbf{x}(s), \hat{\mathbf{v}}(s)) = \int_0^{s(\mathbf{x})} dsn(\mathbf{x}(s)), \tag{C9}$$

where it is understood that $\mathbf{x}$, $\hat{\mathbf{v}}$ depend not only upon s, but also on two other coordinates, say ξ, η, laid out upon the constant s surfaces. So, from (C9),

$$\nabla \mathcal{A}_0(\mathbf{x}) = n(\mathbf{x}(s))\nabla s = n(\mathbf{x}(s))\hat{\mathbf{v}}(s). \tag{C10}$$

This is the same as (C4), so $\Phi_0 = \mathcal{A}_0$.

Φ_0 is called the optical path length. A light ray follows the flow line of the fictitious particle we have been considering but, of course, it moves along that path with the speed of light c/n. So, when a light ray moves through the distance ds, that takes time dt=dsn/c. Thus, according to Eq. (C9), the optical path length Φ_0 is just the integrated time that light takes to go from one place to another, multiplied by c. (The "principle of least time," the idea that the actual path light takes between two points is the path which takes the least time, is due to Fermat in 1662.) As a consequence, all rays of light which have the same phase at the surface $s = 0$ and travel to the surface s have the same phase there. The surface of constant s is called a "wave front."

To complete the WKB approximation, we need to find Φ_1. Setting $\Phi_1 = i\Phi_1^I$ in Eq. (C3), with use of (C4), we have that

$$2n\hat{\mathbf{v}} \cdot \Phi_1^I = 2n\frac{d}{ds}\Phi_1^I = \nabla \cdot (n\hat{\mathbf{v}}) = \frac{d}{ds}n + n\nabla \cdot \hat{\mathbf{v}}.$$

From the second and fourth terms of this equation,

$$\Phi_1^I(\mathbf{x}) = \ln n^{1/2}(\mathbf{x}) + \frac{1}{2}\int_0^{s(\mathbf{x})} ds \nabla \cdot \hat{\mathbf{v}}(\mathbf{x}(s)). \tag{C11}$$

Thus, from Eqs.(C9),(C11), we obtain the WKB approximate solution of the wave equation:

$$U(\mathbf{x}) = n^{-1/2}(\mathbf{x}) e^{-\frac{1}{2}\int_0^{s(\mathbf{x})} ds \nabla \cdot \hat{\mathbf{v}}(\mathbf{x}(s))} e^{ik\int_0^{s(\mathbf{x})} dsn(\mathbf{x}(s))} \tag{C12}$$

Eq. (C12) is what shall be used in what follows. It requires specifying an initial surface for $s = 0$. From this, at any point $\mathbf{x}_0$ on this surface, the initial veocity field $\hat{\mathbf{v}}(\mathbf{x}_0)$ can be determined, since it is perpendicular to the surface and of unit length. Then, one can solve the dynamical equation (C6) to obtain the velocity field elsewhere, and find the specific trajectories $\mathbf{x}(s, \mathbf{x}_0)$. This allows calculation of the integrals in (C12), resulting in the WKB solution $U(\mathbf{x})$. If $n(\mathbf{x}_0) = 1$, this solution has $U(\mathbf{x}_0) = 1$. If a solution with any other value $U_0(\mathbf{x}_0)$ on the $s = 0$ surface is desired, it is $U_0(\mathbf{x}_0)U(\mathbf{x})$.

The last factor in Eq. (C12) is well known in optics, as the eikonal or ray approximation. What has been shown here is that it is justified as the WKB approximate solution of the wave equation.

For our problem, of a point source at $\mathbf{x} = 0$, we choose the $s = 0$ surface to be spherical, of infinitesimal radius, centered at $\mathbf{x} = 0$. Therefore, the initial velocity emanates radially out from $\mathbf{x} = 0$. We assume $n = 1$, for at least a small volume around $\mathbf{x} = 0$. Then, by Eq. (C6), $d\mathbf{v}/dt = 0$ so $\hat{\mathbf{v}}(\mathbf{x}) = \mathbf{r}/r = \hat{\mathbf{r}}$, where $\mathbf{r}$ is the radial vector. Since $\nabla \cdot \hat{\mathbf{r}}f(r) = r^{-2}d^2[r^2 f(r)]/dr^2$, with $f = 1$ we get $\nabla \cdot \hat{\mathbf{v}} = 2/r$. The distance travelled from $s = 0$, along the velocity field, is $s = r$. Putting this into Eq. (C12) gives, in this volume,

$$U(\mathbf{x}) = \frac{1}{r}e^{ikr}. \tag{C13}$$

This satisfies the wave equation Eq. (C1), with a unit point source at the origin.

APPENDIX D: REFLECTION FROM LENSES AND MIRRORS

This subsection is a diversion from our main argument, and may be skipped. It is here for logical completeness, and to make some pedagogical points.

In applications to optical systems, light, initially in vacuum, encounters an abrupt change of index of refraction, in the form of lenses or mirrors. The latter may be accommodated by setting $n = -\infty$ in the volume of the mirror. This may be understood from the quantum theory analogy, where $n = -\infty$ turns the potential well into an infinite potential barrier.

How good is the WKB approximation in this case? For completely empty space, Eq. (C13) is the exact solution of Eq. (C1). However, in non-empty space, there is an obvious failure when two rays cross. In that case, from (C4), $\nabla\Phi_0 \sim \hat{\mathbf{v}}$ would then have two possible values, which is impossible.

1. Mirrors

This is what occurs at the surface of a mirror. For example, for a plane mirror at $z = 0$, and an incoming plane wave of wave number k and direction $\hat{\mathbf{v}} = \hat{\mathbf{j}}a + \hat{\mathbf{k}}b$, we know the solution of the wave equation. It is the sum of incident and reflected waves which vanishes at $z = 0$:

$$U \sim e^{ik(ay+bz)} - e^{ik(ay-bz)} \sim e^{ikay} e^{\ln \sin kbz} \sim e^{ikay + \ln(kbz) - (kbz)^2/6 + \ldots},$$

Obviously, the wave amplitude can no longer be described by a single term of the form $\exp ik\Phi$, where Φ can be expanded in inverse powers of k.

But, the resolution of this difficulty is apparent from the example. It is to find the solution of the wave equation as the *sum* of WKB terms. After all, the wave equation is linear, so a sum of solutions is a solution.

For the case of a finite sized mirror, rays which don't hit the mirror can continue on their merry way. Those incident rays U_{i} which do hit the mirror surface are used to obtain the reflected solution U_{r}. That is, the solution is $U = U_{\text{i}} + U_{\text{r}}$, where U_{r} is to be constructed. This requires, on the mirror surface, knowing U_{r} and the direction of the outgoing vector field $\hat{\mathbf{v}}_{\text{r}}$. These are obtained by requiring the wave equation to be satisfied through the mirror surface, that is, by requiring $U = 0$ and $\nabla_{||} U = 0$ on the mirror surface.

The first condition implies $U_{\text{r}} = -U_{\text{i}}$ at the surface. The second condition implies

$$\nabla_{||} U = ik[U_{\text{i}} \nabla_{||} \Phi_{\text{i}} + U_{\text{r}} \nabla_{||} \Phi_{\text{r}}] = ikU_{\text{i}}[\hat{\mathbf{v}}_{\text{i}||} - \hat{\mathbf{v}}_{\text{r}||}] = 0$$

(the last step uses the WKB approximation of Φ_0 replacing Φ, and (C4)). Thus, $\hat{\mathbf{v}}_{\text{r}||} = \hat{\mathbf{v}}_{\text{i}||}$. Since $\hat{\mathbf{v}}_{\text{r}}$ is a unit vector, this implies $\hat{\mathbf{v}}_{\text{r}\perp} = -\hat{\mathbf{v}}_{\text{i}\perp}$. Thus, the law of reflection is obtained. This completes the specification of the initial conditions for the reflected WKB solution, which can now be constructed using (C12)

The reflected part of U is non-zero within the volume enclosed by the mirror surface and the outermost reflected rays, and abruptly jumps to zero outside. More will be said about this discontinuity at the end of this Appendix.

2. Lenses

A similar situation prevails for a finite sized lens. The WKB approximation's rays travel past or through the lens and beyond. However, even before the WKB solution breaks down (where the focused rays which emerge from the lens eventually cross), something is missing. Light reflects from glass. The single WKB solution does not take that into account.

Accordingly, another solution U_{r} to represent the reflected light must be added. We shall only discuss how to find the light which reflects from the entrance lens surface: light also reflects from the exit lens surface, and that light reflects off the entrance surface, etc: using the method of our discussion, one could do these other calculations if one chose.

Reflected energy means decreased refracted energy. We shall take the refracted solution to be $U_{\text{R}} = AU_{\text{i}}$ within the lens, where $0 < A < 1$ is real. Likewise, within the lens, take $\hat{\mathbf{v}}_{\text{R}}$ to be identical to $\hat{\mathbf{v}}$, the (refracted) continuation through the lens of the incident solution. So, we just need to determine A to complete the specification of U_{R}. In addition, we must find the initial conditions for U_{r} and $\hat{\mathbf{v}}_{\text{r}}$ on the surface, to construct U_{r} elsewhere.

As with the case of a mirror, this information is supplied by requiring the wave equation be satisfied. That is, U and ∇U must be continuous across the lens surface. The first condition implies $U_{\text{i}} + U_{\text{r}} = U_{\text{R}}$ on the surface, i.e., $U_r = -(1 - A)U_{\text{i}}$. The second condition is $U_{\text{i}} ik \nabla \Phi_{\text{i}} + U_{\text{r}} ik \nabla \Phi_{\text{r}} = U_{\text{R}} ik \nabla \Phi_{\text{R}}$ on the surface. With use of Eq. (C4), this gives

$$\hat{\mathbf{v}}_{\text{r}} = \frac{\hat{\mathbf{v}}_{\text{i}} - An\hat{\mathbf{v}}_{\text{R}}}{1 - A}. \tag{D1}$$

We can find A by taking the scalar product of (D1) with itself. Since $\hat{\mathbf{v}}_{\text{i}} \cdot \hat{\mathbf{v}}_{\text{R}} = \cos(\theta_{\text{i}} - \theta_{\text{R}})$, where the angles are those the incident and refracted rays make with the normal to the surface at a point of the surface, we obtain

$$A = \frac{2}{n^2 - 1}[n \cos(\theta_{\text{i}} - \theta_{\text{R}}) - 1]. \tag{D2}$$

Putting (D2) into (D1) and taking (D1)'s scalar product with a unit vector parallel to the surface and in the plane of $\hat{\mathbf{v}}_{\text{i}}$ and $\hat{\mathbf{v}}_{\text{R}}$, results in the law of reflection:

$$\sin \theta_{\text{r}} = \frac{\sin \theta_{\text{i}} - An \sin \theta_{\text{R}}}{1 - A} = \sin \theta_{\text{i}}$$

(using $\sin \theta_{\text{i}} = n \sin \theta_{\text{R}}$, Snell's law). This completes the specification of the initial conditions for the reflected and refracted WKB solutions from the entrance surface of the lens.

For normally incident light, $\theta_{\text{i}} = \theta_{\text{R}}$, it follows from (D2) that $A = 2/(n + 1)$: this is also the result given by electromagnetic theory. Therefore, for $n = 3/2$, the magnitude of the reflected light intensity is $(1 - A)^2 = 1/25$. The intensity of reflected light at any other angle is less than this 4% value. Because it is so small, it shall be unnecessary to consider this reflected solution in subsequent sections.

The purpose of this discussion was not just to show that reflected light can be neglected in considering refracted light through a lens. It was also to emphasize that the sum of WKB solutions is a solution, and that it can be accurate to use the WKB solution up to a surface, and then consider another WKB solution as a continuation of it. Both ideas shall be needed, because something still is missing.

If the lens is backed by an screen containing an aperture (the exit pupil), U_{R} beyond the lens abruptly jumps from its WKB value to zero at the edge of the "ray bundle." Since $\nabla^2 U_{\text{R}}$ is singular there, this cannot satisfy the wave equation. There has to be a modified U_{R} which smooths out this abrupt transition. This brings us to considerations of diffraction.